

ABSTRACT CESÀRO SPACES. II. OPTIMAL RANGE*

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ABSTRACT. Abstract Cesàro spaces are investigated from the optimal domain and optimal range point of view. There is a big difference between the cases on $[0, \infty)$ and on $[0, 1]$, as we can see in Theorem 1. Moreover, we present an improvement of Hardy inequality on $[0, 1]$ which plays an important role in these considerations.

1. INTRODUCTION AND BASIC DEFINITIONS

For a Banach ideal space X on $I = [0, 1]$ or $I = [0, \infty)$ let us consider, as in [LM14], the abstract Cesàro space CX on I defined as $CX = \{f \in L^0(I) : C|f| \in X\}$ with the norm given by

$$\|f\|_{CX} = \|C|f|\|_X,$$

where C is the Cesàro operator

$$Cf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad x \in I.$$

One may look at this space, on the one hand, as on generalizations of the well-known Cesàro spaces $Ces_p[0, 1]$ and $Ces_p[0, \infty)$ which were investigated for example in [AM09] and on the other hand, just by definition, we get $C : CX \rightarrow X$ is bounded and CX is the largest ideal space satisfying this relation, i.e. CX is the optimal domain of C for X . Consequently, the abstract Cesàro spaces may be considered also from the optimal domain point of view, as it was done in [DS07], [NP10], [NP11], [MNS13]. In this paper we discuss the Cesàro function spaces on $[0, \infty)$ and on $[0, 1]$ from the point of view of optimal domain and optimal range of the Cesàro operator C . Such concept was already considered for $X = L^{p(\cdot)}$ on $[0, 1]$ in [NP10], [NP11] and for $X = L^{p(\cdot)}$ on \mathbb{R}^n in [MNS13], although the most interesting situation of CX on $[0, 1]$ was omitted there. We develop and complete the discussion under some minimal assumptions. In this more interesting case of interval $[0, 1]$ a very important role is played by the improvement of Hardy inequality presented in Theorem 2.

We present some basic definitions to understand further description of results. A Banach space $X \subset L^0 = L^0(I)$ is called a Banach ideal space on I if $g \in X$, $f \in L^0(I)$, $|f| \leq |g|$ a.e. on I implies $f \in X$ and $\|f\| \leq \|g\|$. We will also assume that $\text{supp} X = I$, i.e. there exists $f \in X$ with $f(x) > 0$ for each $x \in I$.

For a given Banach ideal space X on I and a function $w \in L^0(I)$ such that $w(x) > 0$ a.e. on I , the weighted ideal Banach space $X(w)$ is defined as $X(w) = \{f \in L^0(I) : fw \in X\}$

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with the norm

$$\|f\|_{X(w)} = \|fw\|_X.$$

In the whole paper only two concrete weights on $I = [0, 1]$ will appear, namely v and $1/v$ where $v(x) = 1 - x$. We will need also a non-increasing majorant \tilde{f} of a given function f , which is just

$$\tilde{f}(x) = \text{ess sup}_{t \in I, t \geq x} |f(t)|, \quad x \in I.$$

Moreover, for a given Banach ideal space X on I , we define a new Banach ideal space $\tilde{X} = \tilde{X}(I)$ as $\tilde{X} = \{f \in L^0(I) : \tilde{f} \in X\}$ with the norm given by

$$\|f\|_{\tilde{X}} = \|\tilde{f}\|_X.$$

By a *symmetric function space* on I with the Lebesgue measure m (symmetric space in short), we mean a Banach ideal space $X = (X, \|\cdot\|_X)$ with the additional property that for any two equimeasurable functions $f \sim g, f, g \in L^0(I)$ (that is, they have the same distribution functions $d_f \equiv d_g$, where $d_f(\lambda) = m(\{x \in I : |f(x)| > \lambda\}), \lambda \geq 0$, and $f \in E$ we have $g \in E$ and $\|f\|_E = \|g\|_E$. In particular, $\|f\|_X = \|f^*\|_X$, where $f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}, t \geq 0$.

The *dilation operators* σ_a ($a > 0$) defined on $L^0(I)$ by

$$\sigma_a f(x) = f(x/a) \chi_I(x/a) = f(x/a) \chi_{[0, \min(1, a)]}(x), \quad x \in I,$$

are bounded in any symmetric space X on I and $\|\sigma_a\|_{X \rightarrow X} \leq \max(1, a)$ (see [BS88, p. 148] and [KPS82, pp. 96-98]). They are also bounded in some Banach ideal spaces which are not necessary symmetric spaces. Furthermore, recall that the Cesàro operator C , the Copson operator C^* and the Hardy-Littlewood maximal operator M are defined, respectively, by

$$Cf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad x \in I, \quad C^*f(x) = \int_{I \cap [x, \infty)} \frac{f(t)}{t} dt, \quad x \in I,$$

$$Mf(x) = \sup_{a, b \in I, 0 \leq a \leq x \leq b} \frac{1}{b-a} \int_a^b |f(t)| dt, \quad x \in I.$$

We refer the reader to [LM14], where basic facts about the spaces CX and \tilde{X} were presented with more details. For more references on Banach ideal spaces and symmetric spaces we refer to [KPS82], [LT79], [BS88], [KA77] and [Ma89].

2. OPTIMAL RANGE

Let X and Y be two Banach ideal spaces on I and let $T : X \rightarrow Y$ be a bounded linear or sublinear operator. A Banach ideal space Z on I is called the *optimal domain* of T for Y within the class of Banach ideal spaces on I , if $T : Z \rightarrow Y$ is bounded and for each Banach ideal space W on I , $T : W \rightarrow Y$ is bounded implies that $W \subset Z$. The last implication may be formulated equivalently as: if Z and W are Banach ideal spaces on I and if $Z \subsetneq W$, then $T : W \not\rightarrow Y$. Of course in such a case $X \subset W$.

Similarly, we shall say that a Banach ideal space Z on I is the *optimal range* of T for X within the class of Banach ideal spaces on I , if $T : X \rightarrow Z$ is bounded and for each Banach ideal space W on I , $T : X \rightarrow W$ is bounded implies that $Z \subset W$. Once again, the last condition may be replaced by: $W \subsetneq Z$ implies $T : X \not\rightarrow W$. Such the optimal range satisfies of course $Z \subset Y$.

The following theorem describes the optimal domain and optimal range problem for Cesàro operator within the class of Banach ideal spaces on I .

Theorem 1. *Let X be a Banach ideal space on I such that the maximal operator M is bounded on X .*

- (i) *If $I = [0, \infty)$, then $C : CX \rightarrow \tilde{X}$ is bounded. Moreover, the space CX is the optimal domain of C for X and for \tilde{X} (also for CX if the dilation operator σ_a is bounded on X for some $0 < a < 1$). The space \tilde{X} is the optimal range of C for CX , X and \tilde{X} . In particular, $CX = C\tilde{X}$.*
- (ii) *If $I = [0, 1]$, then $C : CX \rightarrow \widetilde{X(1/v)(v)}$ is bounded. The space CX is the optimal domain of C for X and also for $\widetilde{X(1/v)(v)}$. Moreover, if the maximal operator M is bounded on X' , then the space $\widetilde{X(1/v)(v)}$ is the optimal range of C for CX and $X(v)$ (cf. Diagram 2). In particular, $CX = C[\widetilde{X(1/v)(v)}]$.*
- (iii) *If $I = [0, 1]$ and the dilation operator $\sigma_{1/2}$ is bounded on X , then $C : C\tilde{X} \rightarrow \tilde{X}$ is bounded. Moreover, the space $C\tilde{X}$ is the optimal domain of C for \tilde{X} and the space \tilde{X} is the optimal range of C for $C\tilde{X}$, X and \tilde{X} . One also has $C\tilde{X} = CX \cap L^1$.*

Before we prove the theorem, let us comment the situation. Suppose that the corresponding assumptions in Theorem 1 are satisfied. Of course, boundedness of M on X implies also boundedness of C on X , therefore support of CX is for sure the same as support of X (cf. [LM14]). Let $I = [0, \infty)$. Then the statement of (i) may be therefore pictured, putting the boundedness of C and respective embeddings, on the following diagram.

$$\begin{array}{ccccc}
 CX & \xrightarrow{C} & \tilde{X} & \hookrightarrow & X & \hookrightarrow & CX \\
 \uparrow & & \nearrow C & & & & \\
 X & & C & & & & \\
 \uparrow & & & & & & \\
 X & & & & & &
 \end{array}$$

Diagram 1

Moreover, point (i) says that, in fact, CX is the optimal domain of C for \tilde{X} , since $CX = C\tilde{X}$. Even more can be said when the dilation operator σ_a is bounded on X for a certain $0 < a < 1$. Then CX is the optimal domain of C even for CX since, by Lemma 6 in [LM14], it follows that $CCX = CX$. On the other hand, we will see that \tilde{X} is the optimal range of C for \tilde{X} , which by the above diagram means that also for X and for CX .

Much more interesting and delicate is the case of interval $[0, 1]$. Suppose that $C : X \rightarrow X$ is bounded and all assumptions of (ii) and (iii) are satisfied. Then $C : CX \rightarrow X$ is bounded, where CX is by definition the optimal domain of C for X . The case (ii) says that the optimal range of C for CX is then $\widetilde{X(1/v)(v)}$. It is however interesting that one

may look at the situation also in another way. Let's start once again with $C : X \rightarrow X$ and find first the optimal range. It appears to be just \widetilde{X} (cf. [NP10, Theorem 8.2], [NP11, Theorem 3.16] and [MNS13, Theorem 4.1]) which is much smaller than $\widetilde{X(1/v)(v)}$. If we now find optimal domain of C for \widetilde{X} it is then just $CX \cap L^1 = C(\widetilde{X})$. The diagram describing this dichotomy is now more complicated.

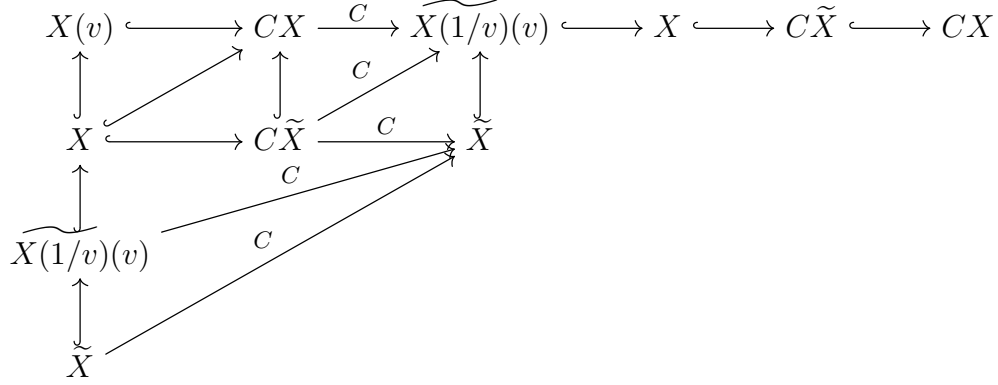


Diagram 2

In general, there is no inclusion relation between $X(v)$ and $C\widetilde{X}$. For example, if X is a symmetric space on $I = [0, 1]$, we have for $f(x) := \frac{1}{1-x}$ that $f \in X(v)$ while $f \notin C\widetilde{X}$ because $Cf(x) \rightarrow \infty$ as $x \rightarrow 1^-$ and so \widetilde{Cf} is not defined (or just ∞ everywhere). Therefore, $X(v) \not\subset C\widetilde{X}$. This means also that C does not act from $X(v)$ into \widetilde{X} . On the other hand, let $X = L^2$ and put $f(x) = |\frac{1}{2} - x|^{-1/2}$. Then $f \notin L^2$, but $Cf \in L^\infty$ and so $\widetilde{Cf} \in L^\infty \subset L^2$. This gives $C\widetilde{X} \not\subset X(v)$. For general symmetric space X on I such that $C : X \rightarrow X$ is bounded, one could take $f \in L^1$ in such a way that $f - f\chi_{[1/2-\epsilon, 1/2+\epsilon]} \in L^\infty$ for each $0 < \epsilon < 1/2$ but $f \notin X$, to achieve the same effect.

Proof of Theorem 1. (ii). Let $0 \leq f \in CX$. Suppose first that $0 \leq y \leq t \leq 2y \leq 1$. Then

$$(2.1) \quad Cf(t) = \frac{1}{t} \int_0^t f(s)ds \geq \frac{1}{2y} \int_0^y f(s)ds = \frac{1}{2}Cf(y).$$

If now $0 \leq x \leq y$ and $y \leq \frac{1}{2}$, then applying (2.1) one gets

$$\begin{aligned} MCf(x) &\geq \frac{1}{2y-x} \int_x^{2y} Cf(t)dt \geq \frac{1}{2y} \int_y^{2y} Cf(t)dt \\ &\geq \frac{1}{2y} \int_y^{2y} \frac{Cf(y)}{2} dt = \frac{1}{4}Cf(y) \geq \frac{1-y}{4(1-x)}Cf(y). \end{aligned}$$

Suppose now that $\frac{1}{2} \leq y \leq t \leq 1$. Then, similarly as in (2.1),

$$(2.2) \quad Cf(t) = \frac{1}{t} \int_0^t f(s)ds \geq \int_0^y f(s)ds \geq \frac{1}{2}Cf(y).$$

In consequence, when $0 \leq x \leq y$ and $\frac{1}{2} \leq y \leq 1$, applying (2.2) we obtain

$$\begin{aligned} M Cf(x) &\geq \frac{1}{1-x} \int_x^1 Cf(t) dt \geq \frac{1}{1-x} \int_y^1 Cf(t) dt \\ &\geq \frac{1}{1-x} \int_y^1 \frac{Cf(y)}{2} dt = \frac{1-y}{2(1-x)} Cf(y). \end{aligned}$$

Consequently,

$$(2.3) \quad M Cf(x) \geq \frac{1}{4(1-x)} \operatorname{ess\,sup}_{0 \leq x \leq y \leq 1} (1-y) Cf(y) = \frac{1}{4(1-x)} \widetilde{[v Cf]}(x).$$

Since M is bounded on X , by our assumption, it follows that

$$\|Cf\|_{\widetilde{X(1/v)}(v)} = \|\widetilde{[v Cf]}/v\|_X \leq 4\|M\|_{X \rightarrow X} \|Cf\|_X = 4\|M\|_{X \rightarrow X} \|f\|_{CX}.$$

This means that $C : CX \rightarrow \widetilde{X(1/v)}(v)$ is bounded and the first statement of (ii) is proved.

It remains to show that the space $\widetilde{X(1/v)}(v)$ is optimal range of C for CX (in fact, even for $X(v)$). Suppose that there is a Banach ideal space Z on I such that

$$Z \subsetneq Y \text{ but } C : CX \rightarrow Z \text{ is bounded.}$$

Let $0 \leq f \in Y \setminus Z$. Define

$$g(x) = \frac{1}{(1-x)} \widetilde{[vf]}(x), x \in I.$$

Then $f \leq g$ and $g \in \widetilde{X(1/v)}(v) \subset X$ because $\frac{1}{1-x} \widetilde{[vg]}(x) = \frac{1}{1-x} \widetilde{[vf]}(x)$. We have

$$\begin{aligned} C(g/v)(x) &= \frac{1}{x} \int_0^x \frac{\widetilde{[vg]}(t)}{(1-t)^2} dt \geq \frac{\widetilde{[vf]}(x)}{x} \int_0^x \frac{1}{(1-t)^2} dt \\ &= \frac{\widetilde{[vf]}(x)}{x} \frac{x}{(1-x)} \geq f(x), \end{aligned}$$

which means that $C(g/v) \notin Z$. However, $g \in X$ and so $g/v \in X(v)$. Also, by Theorem 2, $X(v) \subset CX$ and therefore $g/v \in CX$ which means that $C : CX \not\rightarrow Z$. Note that we have already shown $C : X(v) \not\rightarrow Z$, which by inclusion $X(v) \subset CX$ means that $\widetilde{X(1/v)}(v)$ is the optimal range also for $X(v)$.

(iii). The argument is analogous to the one from statement (5.1) in [NP10]. However, we need to modify it because in [NP10] the maximal operator is defined on a larger interval than $[0, 1]$. Let $0 \leq f \in CX \cap L^1[0, 1]$. We shall understand that $f(x) = 0$ for $x > 1$. Of course, inequality from (2.1) remains true in this case, since $f \in L^1[0, 1]$. Suppose that $0 < x \leq y \leq 1$ and consider two cases. If $y/2 \leq x$, then

$$M\sigma_{1/2} Cf(x) \geq \frac{2}{y} \int_{y/2}^y \sigma_{1/2} Cf(u) du.$$

If $x \leq y/2$, then

$$M\sigma_{1/2} Cf(x) \geq \frac{1}{y-x} \int_x^y \sigma_{1/2} Cf(u) du \geq \frac{1}{y} \int_{y/2}^y \sigma_{1/2} Cf(u) du.$$

Alltogether we get

$$M\sigma_{1/2}Cf(x) \geq \frac{1}{y} \int_{y/2}^y \sigma_{1/2}Cf(u)du = \frac{1}{2y} \int_y^{2y} Cf(t)dt \geq \frac{1}{4}Cf(y).$$

Therefore, similarly as before,

$$M\sigma_{1/2}Cf(x) \geq \frac{1}{4} \operatorname{ess\,sup}_{x \leq y} Cf(y) = \frac{1}{4} \widetilde{C}f(x),$$

which gives

$$\begin{aligned} \|f\|_{C\widetilde{X}} &= \|\widetilde{C}f\|_X \leq 4\|M\sigma_{1/2}Cf\|_X \leq 4\|M\|_{X \rightarrow X}\|\sigma_{1/2}\|_{X \rightarrow X}\|Cf\|_X \\ &= 4\|M\|_{X \rightarrow X}\|\sigma_{1/2}\|_{X \rightarrow X}\|f\|_{CX} \leq 4\|M\|_{X \rightarrow X}\|\sigma_{1/2}\|_{X \rightarrow X}\|f\|_{CX \cap L^1}. \end{aligned}$$

On the other hand, if $0 \leq f \in C\widetilde{X}$, then

$$\|f\|_{L^1} = \int_0^1 f(t)dt \frac{\|\chi_{[0,1]}\|_X}{\|\chi_{[0,1]}\|_X} = \frac{\|(\int_0^1 f(t)dt)\chi_{[0,1]}\|_X}{\|\chi_{[0,1]}\|_X} \leq \frac{\|\widetilde{C}f\|_X}{\|\chi_{[0,1]}\|_X}.$$

Thus also

$$\|f\|_{CX \cap L^1} \leq \max\{1, \frac{1}{\|\chi_{[0,1]}\|_X}\}\|\widetilde{C}f\|_X,$$

which means that $C\widetilde{X} = CX \cap L^1$. For the sake of completeness we present the argument that \widetilde{X} is the optimal range of C for $C\widetilde{X}$, although it works just like in [NP10, Theorem 8.2]. Let Z be a Banach ideal space on I and suppose that $0 \leq f \in \widetilde{X} \setminus Z$. Then also $\widetilde{f} \in \widetilde{X} \setminus Z$ and $C\widetilde{f} \geq \widetilde{f}$. However $\widetilde{f} \notin Z$, which means that $C\widetilde{f} \notin Z$ and $C : C\widetilde{X} \not\rightarrow Z$.

(i) This case is easier and may be deduced directly from [MNS13]. Since for $0 < y$ also $2y \in I$ it is enough to follow (2.1) and after that to get for $y \geq x \geq 0$

$$MCf(x) \geq \frac{1}{2y-x} \int_x^{2y} Cf(t)dt \geq \frac{1}{4}Cf(y).$$

Then

$$\|Cf\|_{\widetilde{X}} = \|\widetilde{C}f\|_X \leq 4\|MCf\|_X \leq 4\|M\|_{X \rightarrow X}\|Cf\|_X = 4\|M\|_{X \rightarrow X}\|f\|_{CX},$$

which means that $C : CX \rightarrow \widetilde{X}$ is bounded and $CX = C\widetilde{X}$. The optimal range of C for \widetilde{X}, X, CX is once again \widetilde{X} and the proof is the same as in (iii) (see also [NP10, Theorem 8.2], [NP11, Theorem 3.16] and [MNS13, Theorem 4.1]). \square

3. HARDY INEQUALITY

We present an improvement of the Hardy inequality which appear for spaces on $I = [0, 1]$.

Theorem 2. *If C is bounded on a Banach ideal space X on $I = [0, 1]$ and maximal operator M is bounded on X' , then*

$$C : X(v) \rightarrow X$$

is also bounded.

Proof. Let $0 \leq f \in X$. We have for $0 < x \leq \frac{1}{2}$

$$C(f/v)(x) = \frac{1}{x} \int_0^x \frac{f(s)}{1-s} ds \leq \frac{2}{x} \int_0^x f(s) ds$$

and for $\frac{1}{2} < x \leq 1$

$$C(f/v)(x) = \frac{1}{x} \int_0^x \frac{f(s)}{1-s} ds \leq 2 \int_0^x \frac{f(s)}{1-s} ds.$$

If we define an operator T as $Tf(x) = \int_0^x \frac{f(s)}{1-s} ds$, then

$$C(f/v) \leq 2(Cf + Tf).$$

Therefore, we need to show that T is bounded on X . Consider an involution operator $\tau : f(x) \mapsto f(1-x)$. Then

$$(3.1) \quad Tf(x) = \int_0^x \frac{f(s)}{1-s} ds = \int_{1-x}^1 \frac{f(1-s)}{s} ds = \tau C^* \tau f(x).$$

Observe that the space

$$X^- = \{f : \tau f \in X\}$$

with its natural norm $\|f\|_{X^-} = \|\tau f\|_X$ is also a Banach ideal space on I and $(X^-)^- = X$. Just by definition $\sigma : X \rightarrow X^-$, $\tau : X^- \rightarrow X$ are bounded and $\tau\tau = id$. Thus T is bounded on X if and only if C^* is bounded on X^- . We will prove the last equivalence. Notice that simply

$$(3.2) \quad Mf(1-x) = \sup_{a \neq b, 0 \leq a \leq 1-x \leq b \leq 1} \frac{1}{b-a} \int_a^b f(s) ds$$

$$(3.3) \quad = \sup_{a \neq b, 0 \leq 1-b \leq x \leq 1-a \leq 1} \frac{1}{b-a} \int_{1-b}^{1-a} f(1-s) ds = (M\tau f)(x)$$

and so $M\tau f = \tau Mf$ which means that for any Banach ideal space Y , M is bounded on Y if and only if M is bounded on Y^- , which by our assumption gives that M is bounded on $(X')^-$. Thus also C is bounded on $(X')^-$ and by duality C^* is bounded on $[(X')^-]'$. However, it is evident that for any Banach ideal space Y there holds $(Y')^- = (Y^-)'$. Then $[(X')^-]' = (X'')^- = X^-$ and so C^* is bounded on X^- . \square

Remark 1. If X is a symmetric space, then evidently $X = X^-$ and we get Lemma 10 from [LM14], which proof was a generalization of the Astashkin - Maligranda result from [AM09]. Moreover, our Theorem 2 includes Theorem 9 in [LM14] for the weighted $L^p(x^\alpha)$ spaces when $1 \leq p < \infty$ and $-1/p < \alpha < 1 - 1/p$.

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